

ON THE TERNARY GOLDBACH PROBLEM WITH PRIMES IN ARITHMETIC PROGRESSIONS OF A COMMON MODULE

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Abstract

For $A, \varepsilon > 0$ and any sufficiently large odd n we show that for almost all $k \leq R := n^{1/5-\varepsilon}$ there exists a representation $n = p_1 + p_2 + p_3$ with primes $p_i \equiv b_i \pmod k$ for almost all admissible triplets b_1, b_2, b_3 of reduced residues mod k .

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1 Introduction and results

Let n be a sufficiently large integer, consider an integer k and let b_1, b_2, b_3 be integers that are relatively prime to $k \geq 1$, we assume that $0 \leq b_i < k$, $i = 1, 2, 3$.

We consider the ternary Goldbach problem of writing n as

$$n = p_1 + p_2 + p_3$$

with primes p_1, p_2 and p_3 satisfying the three congruences

$$p_i \equiv b_i \pmod k, \quad i = 1, 2, 3$$

for the common module k . One is interested in the solvability of this question for all sufficiently large n with the module k being as large as up to some power of n . This problem has been studied intensely by many authors. For an overview, see for example [3].

A necessary condition for solvability is

$$n \equiv b_1 + b_2 + b_3 \pmod k,$$

otherwise no such representation of n is possible.

We call such a triplet b_1, b_2, b_3 of reduced residues mod k *admissible*, and a pair b_1, b_2 of reduced residues *admissible*, if $(n - b_1 - b_2, k) = 1$. For a given b_1 we call b_2 *admissible*, if b_1, b_2 is an admissible pair. Let us denote the number of these admissible pairs respectively triplets by $A(k)$.

We precise our consideration of this strengthened ternary Goldbach problem in the following way. Let

$$J_3(n) := J_{k,b_1,b_2,b_3}(n) := \sum_{\substack{m_1, m_2, m_3 \leq n \\ m_1 + m_2 + m_3 = n \\ m_i \equiv b_i(k), \\ i=1,2,3}} \Lambda(m_1) \Lambda(m_2) \Lambda(m_3),$$

where Λ is von Mangoldt's function. $J_3(n)$ goes closely with the number of representations of n in the way mentioned.

In this paper we prove that the deviation of $J_3(n)$ from its expected main term is uniformly small for large moduli, namely in the following sense.

Theorem 1. *For every $A, \varepsilon > 0$, every sufficiently large n and for $D \leq n^{1/5-\varepsilon}$ it holds that*

$$\mathcal{E} := \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1, k)=1} \frac{1}{\varphi(k)} \sum_{b_2 \text{ adm.}} \left| J_3(n) - \frac{n^2}{k^2} \mathcal{S}(n, k) \right| \ll \frac{n^2}{(\log n)^A}.$$

Here $\mathcal{S}(n, k)$ denotes the singular series for this special Goldbach problem and depends on b_1, b_2 and k likewise $J_3(n)$ does; residue b_3 is simply $b_3 \equiv n - b_1 - b_2 (k)$. Namely, see [2], for odd n we have

$$\mathcal{S}(n, k) = C(k) \prod_{p|k} \frac{p^3}{(p-1)^3 + 1} \prod_{\substack{p|n \\ p \nmid k}} \frac{(p-1)((p-1)^2 - 1)}{(p-1)^3 + 1} \prod_{p>2} \left(1 + \frac{1}{(p-1)^3} \right),$$

where $p > 2$ throughout, $C(k) = 2$ for odd k and $C(k) = 8$ for even k .

As a consequence of Theorem 1, we prove in section 2 the following result.

Theorem 2. *Let $A, \varepsilon > 0$ and let $n \in \mathbb{N}$ be odd and sufficiently large. Then for all $k \leq R := n^{1/5-\varepsilon}$ with at most $\ll R(\log n)^{-A}$ many exceptions of them there exists a representation $n = p_1 + p_2 + p_3$ with primes $p_i \equiv b_i(k)$ for all but $\ll A(k)(\log n)^{-A}$ many admissible triplets b_1, b_2, b_3 .*

So there are few exceptions for k , and also the number of exceptions of admissible triplets is small compared with the number $A(k)$ of all admissible triplets.

Let us compare this Theorem 2 with the result of J. Liu and T. Zhang in [2] who show the assertion for $R := n^{1/8-\varepsilon}$ and *all* admissible triplets. In another paper [4], Z. Cui improved this to $R := n^{1/6-\varepsilon}$. Further C. Bauer and Y. Wang showed in [3] the assertion for $R := n^{5/48-\varepsilon}$, but with only $\ll (\log n)^B$ many exceptions.

Here we improved the bound for R again, but at the cost of possible but few exceptions of admissible triplets.

2 Proof of Theorem 2

First of all we give a lower bound for $A(k)$:

Lemma 1. *For odd n we have $A(k) \gg \frac{\varphi(k)^2}{(\log k)^3}$, more accurate, for every reduced residue $b_1 \bmod k$ there are $\gg \frac{\varphi(k)}{(\log k)^3}$ many reduced residues $b_2 \bmod k$ with $(n - b_1 - b_2, k) = 1$.*

Proof. Fix a reduced residue $b_1 \bmod k$. Now count the b_2 with $(b_2, k) = (n - b_1 - b_2, k) = 1$. So b_2 is to choose such that for all prime divisors $p > 2$ of k we have $b_2 \not\equiv 0 \pmod{p}$ and $b_2 \not\equiv n - b_1 \pmod{p}$, what makes $\geq p - 2$ many possibilities for $b_2 \bmod p$, and $\geq p^{l-1}(p - 2)$ many possibilities for $b_2 \bmod p^l$. If $p = 2$ for even k we have an odd b_1 , so $n - b_1$ is even and therefore one can take $b_2 \equiv 1(2)$, so there are $2^{\nu_2(k)-1}$ many possibilities for $b_2 \bmod 2^{\nu_2(k)}$, if $2^{\nu_2(k)} \parallel k$.

Therefore the number of b_2 is at least

$$2^{\max\{0, \nu_2(k)-1\}} \prod_{\substack{p^l \parallel k \\ p \neq 2}} p^{l-1}(p - 2) = \varphi(k) \prod_{\substack{p \mid k \\ p \neq 2}} \frac{p - 2}{p - 1}$$

with

$$\prod_{\substack{p \mid k \\ p \neq 2}} \frac{p - 1}{p - 2} = \prod_{\substack{p \mid k \\ p \neq 2}} \left(1 + \frac{1}{p - 2}\right) \leq \prod_{p \mid k} \left(1 + \frac{2}{p - 1}\right)$$

$$\leq \sum_{q=1}^k \frac{\mu(q)^2 2^{\omega(q)}}{\varphi(q)} \ll \sum_{q=1}^k \frac{\tau(q)}{q} \log k \ll (\log k)^3.$$

This shows the Lemma. \square

Now we show Theorem 2 as a corollary of Theorem 1.

Fix $A, \varepsilon > 0$ and let n be odd and sufficiently large. Consider

$$R_3(n) := \sum_{\substack{p_1, p_2, p_3 \\ p_1 + p_2 + p_3 = n \\ p_i \equiv b_i(k), \\ i=1,2,3}} \log p_1 \log p_2 \log p_3 \quad \text{and} \quad r_3(n) := \sum_{\substack{p_1, p_2, p_3 \\ p_1 + p_2 + p_3 = n \\ p_i \equiv b_i(k), \\ i=1,2,3}} 1.$$

Let $D < k \leq 2D$ with $D \leq R := n^{1/5-\varepsilon}$. For any admissible triplet $b_1, b_2, b_3 \pmod k$ we have

$$|R_3(n) - J_3(n)| \leq (\log n)^3 W_3,$$

where W_3 denotes the number of solutions of $p^l + q^j + r^m = n$ with p, q, r prime, and where l, j or m are at least 2 such that $p^l \equiv b_1(k)$, $q^j \equiv b_2(k)$ and $r^m \equiv b_3(k)$.

Now we prove that

$$\sum_{D < k \leq 2D} k \max_{\substack{b_1, b_2, b_3 \\ \text{admissible}}} W_3 \ll \frac{n^2}{(\log n)^{A+3}}.$$

For this, we split the number W_3 according to if at least two of the exponents l, j, m are ≥ 2 or only one, and for this we write $W_3 = W_1 + W_2$. There are at most \sqrt{n} prime powers $\leq n$ with exponent ≥ 2 , so in the first case we have $W_1 \ll n$, and the left hand side with W_1 is $\ll D^2 W_1 \ll D^2 n \ll \frac{n^2}{(\log n)^{A+3}}$.

In the second case, if only one exponent is ≥ 2 , we have $W_2 \ll \sqrt{n} \cdot \frac{n}{k} = \frac{n^{3/2}}{k}$, and so the left hand side is $\ll D n^{3/2} \ll \frac{n^2}{(\log n)^{A+3}}$.

So for $D \leq n^{1/5-\varepsilon}$ it follows from Theorem 1:

$$\sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1, k)=1} \frac{1}{\varphi(k)} \sum_{b_2 \text{ adm.}} \left| R_3(n) - \frac{n^2}{k^2} \mathcal{S}(n, k) \right|$$

$$\begin{aligned}
&\leq \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1, k)=1} \frac{1}{\varphi(k)} \sum_{b_2 \text{ adm.}} \left| R_3(n) - J_3(n) \right| \\
&\quad + \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1, k)=1} \frac{1}{\varphi(k)} \sum_{b_2 \text{ adm.}} \left| J_3(n) - \frac{n^2}{k^2} \mathcal{S}(n, k) \right| \\
&\ll (\log n)^3 \sum_{D < k \leq 2D} k \max_{\substack{b_1, b_2, b_3 \\ \text{admissible}}} W_3 + \frac{n^2}{(\log n)^A} \ll \frac{n^2}{(\log n)^A}.
\end{aligned}$$

So the formula of Theorem 1 holds also for $R_3(n)$ instead of $J_3(n)$.

Now for $D < k \leq 2D$ we have $A(k) := \#\{b_1, b_2 \text{ admissible mod } k\}$, and let $T(k) := \#\{b_1, b_2 \text{ admissible mod } k; R_3(n) = 0\}$ and consider the set

$$\mathcal{K}_D := \{k; D < k \leq 2D, T(k) \geq A(k)(\log n)^{-A}\}$$

and let K_D be its number.

Since $\mathcal{S}(n, k) \gg 1$ if it is positive, what is the case for admissible triplets and odd n (see its formula above as an Euler product), we have

$$\begin{aligned}
K_D \cdot \frac{n^2}{D} &\ll \sum_{\substack{D < k \leq 2D \\ k \in \mathcal{K}_D}} \frac{k}{T(k)} \sum_{\substack{b_1, b_2 \text{ adm.} \\ R_3(n)=0}} \left| \frac{n^2}{k^2} \mathcal{S}(n, k) \right| \\
&\ll \sum_{D < k \leq 2D} \frac{k}{A(k)} \sum_{b_1, b_2 \text{ adm.}} (\log n)^A \left| R_3(n) - \frac{n^2}{k^2} \mathcal{S}(n, k) \right| \\
&\ll (\log n)^{A+3} \sum_{D < k \leq 2D} \frac{k}{\varphi(k)^2} \sum_{\substack{b_1, b_2 \\ \text{adm.}}} \left| R_3(n) - \frac{n^2}{k^2} \mathcal{S}(n, k) \right| \ll \frac{n^2}{(\log n)^A},
\end{aligned}$$

using Lemma 1 and the above. Therefore it follows that $K_D \ll D(\log n)^{-A}$, so for all $k \notin \mathcal{K}_D$ we have $R_3(n) > 0$ for all but $\ll A(k)(\log n)^{-A}$ many admissible triplets b_1, b_2, b_3 , and then $r_3(n) \gg R_3(n)(\log n)^{-3}$ is positive, too. This shows Theorem 2, since the overall number of exceptions is

$$\ll \sum_{i \ll \log R} \#\mathcal{K}_{2^i} \ll (\log n) \cdot \frac{R}{(\log n)^{A+1}} = \frac{R}{(\log n)^A}.$$

□

3 Proof of Theorem 1

We are going to show Theorem 1 in two steps according to the circle method.

Let $A, \varepsilon, \theta > 0$, $B \geq 2A + 1$ and $D \leq n^{1/4}(\log n)^{-\theta}$.

We define major arcs $\mathfrak{M} \subseteq \mathbb{R}$ by

$$\mathfrak{M} := \bigcup_{q \leq D(\log n)^B} \bigcup_{\substack{0 < a < q \\ (a, q) = 1}} \left[\frac{a}{q} - \frac{D(\log n)^B}{qn}, \frac{a}{q} + \frac{D(\log n)^B}{qn} \right]$$

and minor arcs by

$$\mathfrak{m} := \left[-\frac{D(\log n)^B}{n}, 1 - \frac{D(\log n)^B}{n} \right] \setminus \mathfrak{M}.$$

For $\alpha \in \mathbb{R}$ and some residue $b \bmod k$ denote

$$S_b(\alpha) := S_{b,k}(\alpha) := \sum_{\substack{m \leq n \\ m \equiv b \pmod{k}}} \Lambda(m) e(\alpha m).$$

From the orthogonal relations for $e(\alpha m)$ it follows that

$$J_3(n) = \int_0^1 S_{b_1}(\alpha) S_{b_2}(\alpha) S_{b_3}(\alpha) e(-n\alpha) d\alpha.$$

By

$$J_3^{\mathfrak{M}}(n) := \int_{\mathfrak{M}} S_{b_1}(\alpha) S_{b_2}(\alpha) S_{b_3}(\alpha) e(-n\alpha) d\alpha$$

we denote the value of the integral for $J_3(n)$ on the major arcs \mathfrak{M} and by

$$J_3^{\mathfrak{m}}(n) := J_3(n) - J_3^{\mathfrak{M}}(n)$$

the value on the minor arcs \mathfrak{m} .

Concerning the major arcs we have

Theorem 3. *For $D \leq n^{1/5-\varepsilon}$ it holds that*

$$\mathcal{E}^{\mathfrak{M}} := \sum_{D < k \leq 2D} k \max_{\substack{b_1, b_2, b_3 \\ \text{admissible}}} \left| J_3^{\mathfrak{M}}(n) - \frac{n^2}{k^2} \mathcal{S}(n, k) \right| \ll \frac{n^2}{(\log n)^A}.$$

We can give the proof of Theorem 3 very shortly, as it is simply done by adapting the result of J. Liu and T. Zhang in [2] for the here given major arcs. In fact, by pursuing their proof we see that for $P := D(\log n)^B$ and $Q := \frac{n}{D(\log n)^B}$ and any $U \leq P$, we have to choose D such that the conditions

$$\begin{aligned} U &\leq n^{1/3}(\log n)^{-E}, & (UQ)^{-1} &\leq U^{-3}(\log n)^{-E} \\ DU &\leq D^{1/3-\delta}n^{1/3}(\log n)^{-E}, & (UQ)^{-1} &\leq D^{1-\delta}(DU)^{-3}(\log n)^{-E} \end{aligned}$$

are satisfied for any $E > 0$ and small $\delta > 0$. The optimal choice of D is therefore given by $D \leq n^{1/5-\varepsilon}$ what proves Theorem 3. The improvement in this paper comes from the different intervals given as major and minor arcs such that dealing on the minor arcs with mean values over b_1, b_2 is still possible.

Namely, as estimate on the minor arcs we show in the next section 4:

Theorem 4. *For $D \leq n^{1/4}(\log n)^{-\theta}$ we have*

$$\mathcal{E}^{\mathfrak{m}} := \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1, k)=1} \frac{1}{\varphi(k)} \sum_{\substack{(b_2, k)=1 \\ \text{adm.}}} |J_3^{\mathfrak{m}}(n)| \ll \frac{n^2}{(\log n)^A}.$$

Theorem 1 is then a corollary of Theorems 3 and 4 since $\mathcal{E} \leq \mathcal{E}^{\mathfrak{M}} + \mathcal{E}^{\mathfrak{m}}$.

This Theorem 4 is the interesting part of Theorem 1, where we can gain a higher power of n for the bound of D by considering the mean value over b_1, b_2 instead of the maximum. But due to this we have to allow exceptions of admissible triplets in Theorem 2, as we have seen in its proof.

In both Theorems 3 and 4 the resulting bound for D is the optimum with the given method, these bounds cannot be balanced to get a larger range than $n^{1/5}$. Also the cited method for the major arcs cannot be improved to gain from mean values over b_1, b_2 since the used character sum estimates are independent of b_1, b_2 . But it may be possible that another method would succeed on \mathfrak{M} .

4 Proof of Theorem 4, the estimate on the minor arcs

Let $D \leq n^{1/4}(\log n)^{-\theta}$ and consider $\mathcal{E}^{\mathfrak{m}}$, it is (where b_1, b_2 run through all reduced residues mod k if indicated by a star)

$$\begin{aligned}
&\ll \sum_{D < k \leq 2D} \frac{k}{\varphi(k)^2} \sum_{b_1, b_2}^* |J_3^{\mathfrak{m}}(n)| \\
&\leq \sum_{D < k \leq 2D} \frac{k}{\varphi(k)^2} \sum_{b_1, b_2}^* \int_{\mathfrak{m}} |S_{b_1}(\alpha) S_{b_2}(\alpha) S_{n-b_1-b_2}(\alpha)| d\alpha \\
&= \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{b_1}^* \int_{\mathfrak{m}} |S_{b_1}(\alpha)| \cdot \frac{1}{\varphi(k)} \sum_{b_2}^* |S_{b_2}(\alpha) S_{n-b_1-b_2}(\alpha)| d\alpha \\
&\leq \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \int_{\mathfrak{m}} \sum_{b_1}^* |S_{b_1}(\alpha)| \\
&\quad \cdot \frac{1}{\varphi(k)} \left(\sum_{b_2 \bmod k} |S_{b_2}(\alpha)|^2 \right)^{1/2} \left(\sum_{b_2 \bmod k} |S_{n-b_1-b_2}(\alpha)|^2 \right)^{1/2} d\alpha \\
&\leq \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \max_{\alpha \in \mathfrak{m}} \sum_{b_1}^* |S_{b_1}(\alpha)| \frac{1}{\varphi(k)} \sum_{b_2 \bmod k} \int_0^1 |S_{b_2}(\beta)|^2 d\beta \\
&\ll n(\log n)^3 \sum_{D < k \leq 2D} \frac{1}{\varphi(k)} \max_{\alpha \in \mathfrak{m}} \sum_{b_1}^* |S_{b_1}(\alpha)| \\
&\leq n(\log n)^3 \sum_{D < k \leq 2D} \max_{\alpha \in \mathfrak{m}} \left(\frac{1}{\varphi(k)} \sum_{b_1}^* |S_{b_1}(\alpha)|^2 \right)^{1/2} \\
&\ll n(\log n)^3 \sum_{D < k \leq 2D} \left(\frac{n^2}{D^2 (\log n)^{2A+6}} \right)^{1/2}.
\end{aligned}$$

In the last step we use Lemma 2 that will be shown next, valid for $D \leq n^{1/4}(\log n)^{-\theta}$ and suitable chosen $\theta, B > 0$ depending just on $A > 0$.

Now the above is $\ll n(\log n)^3 D \frac{n}{D(\log n)^{A+3}} \ll \frac{n^2}{(\log n)^A}$ as was to be shown for the minor arcs. \square

So what is left is to show

Lemma 2. *For all $A > 0$ and $B \geq 2A + 1, \theta \geq B/2$ let $D \leq n^{1/4}(\log n)^{-\theta}$ and $\alpha \in \mathbb{R}$ with $\|\alpha - \frac{u}{v}\| < \frac{1}{v^2}$ for some integers u, v with $(u, v) = 1$ and*

$D(\log n)^B \leq v \leq \frac{n}{D(\log n)^B}$. Then for $D < d \leq 2D$ we have

$$\frac{1}{\varphi(d)} \sum_{c, (c,d)=1} |S_{c,d}(\alpha)|^2 \ll \frac{n^2}{D^2(\log n)^A}.$$

We remark that for $\alpha \in \mathfrak{m}$ there exist u, v with $(u, v) = 1$, $v \leq \frac{n}{D(\log n)^B}$ and $|\alpha - \frac{u}{v}| < \frac{D(\log n)^B}{vn} \leq \frac{1}{v^2}$ by Dirichlet's approximation theorem, so $v \geq D(\log n)^B$ since $\alpha \in \mathfrak{m}$, and therefore the conditions of Lemma 2 are fulfilled.

For the proof we need the following well known auxiliary Lemma.

Lemma 3. Let $|\alpha - \frac{u}{v}| \leq \frac{1}{v^2}$, $(u, v) = 1$. Then

$$\sum_{m \leq X} \min(Y, |\alpha m|^{-1}) \ll \frac{XY}{v} + (X + v)(\log v).$$

Proof of Lemma 2. Fix n large and $D \leq n^{1/4}(\log n)^{-\theta}$, and let α , u and v be as given in Lemma 2.

We apply Vaughan's identity on the exponential sum $S_{c,d}(\alpha)$, see for example A. Balog in [1], where a similar Lemma is shown (Lemma 2 there). From that it follows that it suffices to show for any complex coefficients $|a_m|, |b_k| \leq 1$ and any $M \in \mathbb{N}$ with

$$\begin{aligned} I : & \quad M \leq V^2, \text{ if } b_k = 1 \text{ for all } k, \\ II : & \quad V \leq M \leq \frac{n}{V} \text{ else, where } V := D(\log n)^B, \end{aligned}$$

we have

$$\sum_{(c,d)=1} \left| \sum_{m \sim M} \sum_{\substack{k \leq n/m \\ km \equiv c(d)}} a_m b_k e(\alpha mk) \right|^2 \ll \frac{n^2}{D(\log n)^A}.$$

Here $m \sim M$ means $M < m \leq M'$ for some $M' \leq 2M$.

We consider first **case II**: Then the left hand side becomes (where m^* denotes the inverse of $m \bmod d$):

$$II := \sum_{(c,d)=1} \left| \sum_{\substack{m \sim M \\ (m,d)=1}} a_m \sum_{\substack{k \leq n/m \\ km \equiv c(d)}} b_k e(\alpha mk) \right|^2$$

$$\begin{aligned}
&\leq \sum_{(c,d)=1} M \sum_{\substack{m \sim M \\ (m,d)=1}} \left| \sum_{\substack{k \leq n/m \\ k \equiv cm^*(d)}} b_k e(\alpha m k) \right|^2 \\
&= M \sum_{\substack{m \sim M \\ (m,d)=1}} \sum_{\substack{(c,d)=1 \\ k \leq n/m \\ k \equiv c(d)}} \left| \sum b_k e(\alpha m k) \right|^2 \\
&= M \sum_{m \sim M} \sum_{\substack{(c,d)=1 \\ k \leq n/m \\ k \equiv c(d)}} \sum_{\substack{k' \leq n/m \\ k' \equiv k(d)}} b_k \overline{b_{k'}} e(\alpha m(k - k')) \\
&= M \sum_{m \sim M} \sum_{\substack{k \leq n/m \\ (k,d)=1}} b_k \sum_{\substack{k' \leq n/m \\ k' \equiv k(d)}} \overline{b_{k'}} e(\alpha m(k - k')) \\
&= M \sum_{m \sim M} \sum_{\substack{k \leq n/m \\ (k,d)=1}} b_k \sum_{\substack{l \geq (k-n/m)/d \\ l \leq (n/m-1)/d}} \overline{b_{k-ld}} e(\alpha m l d) \\
&\leq M \sum_{k \leq n/M} \sum_{|l| \leq n/Md} \left| \sum_{\substack{m \sim M \\ m \leq n/k \\ m \leq n/\max\{k-ld, ld+1\}}} e(\alpha m l d) \right|.
\end{aligned}$$

Now the exponential sum in absolute value is $\ll \min(M, \|\alpha l d\|^{-1})$, so the estimation goes on with

$$\begin{aligned}
&\ll M \frac{n}{M} \sum_{|l| \leq n/Md} \min(M, \|\alpha l d\|^{-1}) \\
&\ll n \sum_{\substack{L \leq n/M \\ d|L}} \min(M, \|\alpha L\|^{-1}) + nM \\
&\leq n \left(\sum_{\substack{L \leq n/M \\ d|L}} 1^2 \right)^{1/2} \left(\sum_{L \leq n/M} M \min(M, \|\alpha L\|^{-1}) \right)^{1/2} + nM \\
&\ll n \left(\frac{n}{Md} \right)^{1/2} M^{1/2} \left(\frac{n}{v} + \left(\frac{n}{M} + v \right) (\log n) \right)^{1/2} + nM
\end{aligned}$$

because of the auxiliary Lemma 3. So expression II is $\ll \frac{n^2}{D(\log n)^A}$ since we have $D(\log n)^B = V \leq M \leq n/V$ in case II, and since $D(\log n)^B \ll v \ll \frac{n}{D(\log n)^B}$ for $B \geq 2A + 1$.

Now consider **case I**: Then the left hand side becomes (where m^* denotes the inverse of $m \bmod d$):

$$\begin{aligned}
I &:= \sum_{(c,d)=1} \left| \sum_{\substack{m \sim M \\ (m,d)=1}} a_m \sum_{\substack{k \leq n/m \\ km \equiv c(d)}} e(\alpha mk) \right|^2 \\
&\leq \sum_{(c,d)=1} M \sum_{\substack{m \sim M \\ (m,d)=1}} \left| \sum_{\substack{k \leq n/m \\ k \equiv cm^*(d)}} e(\alpha mk) \right|^2 \\
&\leq M \sum_{m \sim M} \sum_{(c,d)=1} \left| \sum_{\substack{k \leq n/m \\ k \equiv c(d)}} e(\alpha mk) \right|^2 \\
&= M \sum_{m \sim M} \sum_{(c,d)=1} \sum_{\substack{k \leq n/m \\ k \equiv c(d)}} e(\alpha mk) \sum_{\substack{k' \leq n/m \\ k' \equiv c'(d)}} e(-\alpha mk') \\
&= M \sum_{m \sim M} \sum_{\substack{k \leq n/m \\ (k,d)=1}} \sum_{\substack{k' \leq n/m \\ k' \equiv k'(d)}} e(\alpha m(k - k')) \\
&\leq M \sum_{m \sim M} \sum_{k \leq n/m} \left| \sum_{\substack{l \geq (k-n/m)/d \\ l \leq (n/m-1)/d}} e(\alpha mdl) \right| \\
&\ll M \sum_{m \sim M} \sum_{k \leq n/M} \left(\min \left(\frac{n}{Md}, \|\alpha md\|^{-1} \right) + 1 \right) \\
&\ll n \sum_{m \sim M} \min \left(\frac{n}{Md}, \|\alpha md\|^{-1} \right) + Mn \\
&\ll n \sum_{\substack{L \sim Md \\ d|L}} \min \left(\frac{n}{Md}, \|\alpha L\|^{-1} \right) + Mn \\
&\ll n \left(\sum_{\substack{L \sim Md \\ d|L}} 1 \right)^{1/2} \left(\sum_{L \sim Md} \frac{n}{Md} \min \left(\frac{n}{Md}, \|\alpha L\|^{-1} \right) \right)^{1/2} + Mn \\
&\ll n \left(\frac{Md}{d} \right)^{1/2} \left(\frac{n}{Md} \right)^{1/2} \left(\frac{n}{v} + (Md + v)(\log n) \right)^{1/2} + Mn
\end{aligned}$$

using again the auxiliary Lemma 3. Now we get $I \ll \frac{n^2}{D(\log n)^A}$ since $D(\log n)^B \ll$

$v \ll \frac{n}{D(\log n)^B}$ with $B \geq 2A + 1$ and since $Md \ll V^2d \ll D^3(\log n)^B \ll \frac{n}{D(\log n)^B}$ for $D \leq n^{1/4}(\log n)^{-\theta}$ and $\theta \geq B/2$. So Lemma 2 is shown. \square

Remark added by author. As was kindly pointed out to me by Z. Cui, it is possible to improve the statement on the major arcs such that Theorems 1, 2 and 3 hold for the improved exponent $1/4$ instead of $1/5$. This major arc improvement has its idea in the publication of Z. Cui in [4].

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